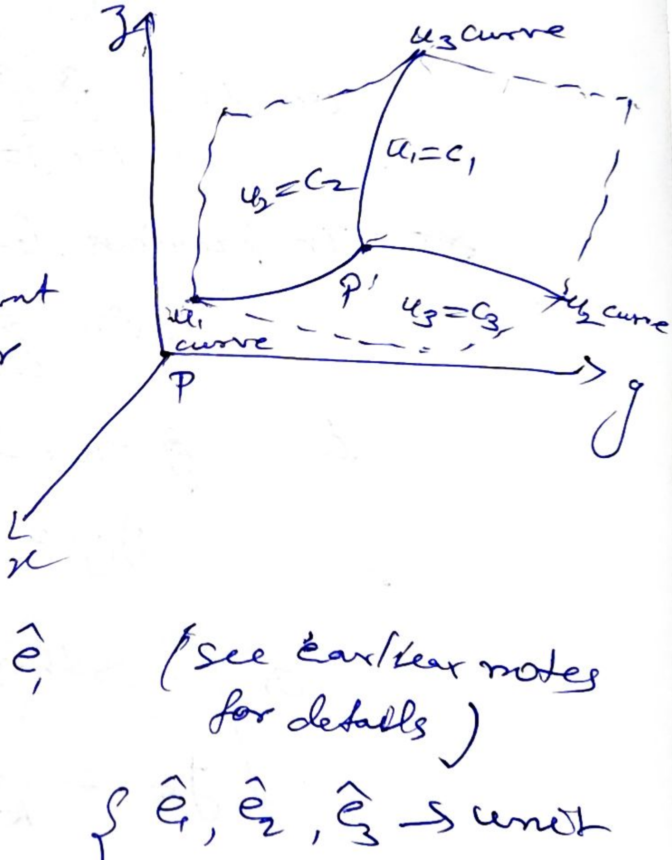
Curvilinear Coordinates Continued---

In the earliar lecture notes we have discussed general curvilinear coordinates, orthogonal curvilinear coordinates, metric coefficients, scale factors etc. Here we discuss area & volume elements, gradients ~~For a positi~~ divergence and ~~curve~~ curl in curvilinear coordinates.

Area & volume elements:

For a position vector $\vec{r} = xi + yj + zk$ of a point P

$\vec{r} = \vec{r}(u_1, u_2, u_3)$.



Unit vector can be defined as

$$\hat{e}_1 = \frac{\frac{\partial \vec{r}}{\partial u_1}}{\left|\frac{\partial \vec{r}}{\partial u_1}\right|}, \quad \frac{\partial \vec{r}}{\partial u_1} \rightarrow \text{tangent vector to the curve } u_1 \text{ at } P.$$

or $$\frac{\partial \vec{r}}{\partial u_1} = \left|\frac{\partial \vec{r}}{\partial u_1}\right| \hat{e}_1 = h_1 \hat{e}_1$$ (see earlier notes for details)

Similary $$\frac{\partial \vec{r}}{\partial u_2} = h_2 \hat{e}_2$$ { $\hat{e}_1, \hat{e}_2, \hat{e}_3 \rightarrow$ unit vectors }

$$\frac{\partial \vec{r}}{\partial u_3} = h_3 \hat{e}_3$$

For $\vec{r} = \vec{r}(u_1, u_2, u_3)$, $d\vec{r}$ can be written as

$$d\vec{r} = \frac{\partial r}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

or $\boxed{d\vec{r} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3}$

and $ds^2 = d\vec{r} \cdot d\vec{r}$

or $\boxed{ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2}$

or in compact form area element can be written as

$\boxed{d\sigma_{ij} = h_i h_j \, du_i du_j}$

Volume element of an orthogonal curvilinear coordinates is given by

$$d\tau = |(h_1 du_1 \hat{e}_1) \cdot (h_2 du_2 \hat{e}_2) \times (h_3 du_3 \hat{e}_3)|$$

or $\boxed{d\tau = h_1 h_2 h_3 \, du_1 du_2 du_3}$ $\left\{ \begin{array}{l} \text{Here} \\ |\hat{e}_1 \cdot \hat{e}_2 \times \hat{e}_3| = 1 \end{array} \right.$

Gradient, divergence and curl in Orthogonal Curv. Coord.:

- Let $\phi \to$ Scalar function
  $\vec{A} \to$ Vector function
  $\vec{A} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3$

$\left\{ \begin{array}{l} \phi = \phi(u_1, u_2, u_3) \\ A_1 = A_1(u_1, u_2, u_3) \\ A_2 = A_2(u_1, u_2, u_3) \\ A_3 = A_3(u_1, u_2, u_3) \end{array} \right.$

(A) Gradient — $\nabla\phi = \text{grad}\,\phi = \hat{e}_1 \frac{\partial \phi}{\partial s_1} + \hat{e}_2 \frac{\partial \phi}{\partial s_2} + \hat{e}_3 \frac{\partial \phi}{\partial s_3}$

or $\boxed{\nabla\phi = \frac{\hat{e}_1}{h_1} \frac{\partial \phi}{\partial u_1} + \frac{\hat{e}_2}{h_2} \frac{\partial \phi}{\partial u_2} + \frac{\hat{e}_3}{h_3} \frac{\partial \phi}{\partial u_3}}$

(B) $\nabla \cdot \vec{A} = \text{div}\vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1}(h_2 h_3 a_1) + \frac{\partial}{\partial u_2}(h_3 h_1 a_2) + \frac{\partial}{\partial u_3}(h_1 h_2 a_3) \right]$

(C) Curl $\to \nabla \times \vec{A} = \text{Curl}\,\vec{A} = \frac{1}{h_1 h_2 h_3} \begin{bmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 a_1 & h_2 a_2 & h_3 a_3 \end{bmatrix}$

(Prove: H.W.)